Full exceptional collections and stability conditions for Dynkin quivers.

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§1. stability condition

$\mathcal{D}$: $\mathbb{C}$-linear tri. cat. of finite type.

$K_0(\mathcal{D})$: Grothendieck grp. of $\mathcal{D}$.

$\mathcal{A} \subset \mathcal{D}$: a heart of a bounded t-str.

$\rightsquigarrow$ $\mathcal{A}$ is an abelian cat.

$\rightsquigarrow$ $K_0(\mathcal{A}) \xrightarrow{\sim} K_0(\mathcal{D})$.

Def (Bridgeland)

(i) A grp hom $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ is a stability function on $\mathcal{A}$.

$\overset{\text{def.}}{\iff}$ $\forall E_{\neq 0} \in \mathcal{A}$, $Z(E) \in \mathbb{H}_- := \left\{ re^{i\pi\phi} \;\middle|\; \begin{matrix} r>0 \\ 0<\phi\leq 1 \end{matrix} \right\}$.

(ii) $(Z, \mathcal{A})$ is a stability condition on $\mathcal{D}$.

$\overset{\text{def.}}{\Longleftrightarrow}$ · $\mathcal{A}$: heart in $\mathcal{D}$

· $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$: stab. func.

Z satisfies "HN-prop." & "support prop.".

$\sigma = (Z, \mathcal{A})$: stab. cond.

$$\phi(E) := \frac{1}{\pi} \operatorname{Arg} Z(E) \in (0, 1].$$

E is σ-semistable

(resp. σ-stable).

$\overset{\text{def.}}{\Longleftrightarrow}$ $0 \neq F \subset E \Rightarrow \phi(F) \leq \phi(E)$

(resp. $\phi(F) < \phi(E)$).

$\operatorname{Stab}(\mathcal{D}) := \{ (Z, \mathcal{A}) : \text{stab. cond. on } \mathcal{D} \}$.

$\exists$ topology on $\operatorname{Stab}(\mathcal{D})$.

Thm (Bridgeland)

$$\begin{array}{ccc} \mathrm{Stab}(\mathcal{D}) & \longrightarrow & \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (Z, \mathcal{A}) & \longmapsto & Z \end{array}$$

is a local homeo.

In particular, $\exists$ cpx. str. on $\mathrm{Stab}(\mathcal{D})$.

§2. exceptional collection and stability condition

Def.

(i) $E \in \mathcal{D}$ is *exceptional*

$\overset{\text{def.}}{\Longleftrightarrow}$ $\mathrm{Hom}(E, E[p]) \cong \begin{cases} \mathbb{C}, & p = 0, \\ 0, & p \neq 0. \end{cases}$

(ii) $(E_1, \cdots, E_\mu)$ is an *exceptional collection*

$\overset{\text{def.}}{\Longleftrightarrow}$ ① $E_1, \cdots, E_\mu$ are exc. obj.s.

② $\mathrm{Hom}(E_j, E_i[p]) \cong 0$ for $j > i$, $p \in \mathbb{Z}$.

(iii) An exc. coll. $(E_1, \cdots, E_\mu)$ is full

$\overset{\text{def.}}{\iff}$ $\left\{\begin{array}{l}\text{the smallest tri. cat.} \\ \text{containing } E_1, \cdots, E_\mu\end{array}\right\} \simeq \mathcal{D}$.

Remark

$\mathcal{A}$: abelian cat. $\quad E, F \in \mathcal{A}$

$\Rightarrow \quad \mathrm{Ext}^p_{\mathcal{A}}(E, F) \simeq \mathrm{Hom}_{D^b(\mathcal{A})}(E, F[p])$.

Def.

An exc. coll. $(E_1, \cdots, E_\mu)$ is Ext.

$\overset{\text{def.}}{\iff}$ $\mathrm{Hom}(E_i, E_j[p]) \simeq 0 \quad , \quad p \leq 0$.

e.g.

Q : acyclic quiver $\quad Q_0 = \{1, \cdots, \mu\}$.

$$j > i \implies \#\{j \to i\} = 0 .$$

S_i : simple left $\mathbb{C}Q$-mod corresp. to $i \in Q_0$.

$\Rightarrow$ $(S_1, \cdots, S_\mu)$ is a full Ext-exc. coll.

s.t.

$$\dim_{\mathbb{C}} \mathrm{Ext}^1(S_i, S_j) = \#\{i \to j\}.$$

<u>Prop</u> (Macrì)

$\mathcal{E} = (E_1, \cdots, E_\mu)$ is a full Ext-exc. coll.

$\Rightarrow$ $\langle \mathcal{E} \rangle_{ex}$ is a heart of a bdd t-str.

extension-closed sub cat.

$\mathcal{E}$: full Ext-exc. coll.

Simple obj.s in $\langle \mathcal{E} \rangle_{ex} = \{E_1, \cdots, E_\mu\}$.

In particul, $\langle \mathcal{E} \rangle_{ex}$ is of finite length.

(i.e.,

$^\forall E \neq 0$ in $\langle \mathcal{E} \rangle_{ex}$, $^\exists$Jordan-Hölder filtration.

A stability function $Z : K_0(\mathcal{A}) \longrightarrow \mathbb{C}$ satisfies the Harder – Narasimhan prop.

If $\mathcal{D} = D^b(\vec{\Delta})$ for a Dynkin quiver $\vec{\Delta}$, any stab. func. satisfies the support prop.

Def

$\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(\mathcal{D})$.

An exc. coll $\mathcal{E} = (E_1, \cdots, E_\mu)$ is a σ-exc. coll.

$\overset{\text{def.}}{\iff}$ ① $E_1, \cdots, E_\mu$ are σ-semistable.

② $\mathcal{E}$ is an Ext-exc. coll.

③ ${}^\exists r \in \mathbb{R}$ s.t. $r < \phi(E_i) \leqq r+1$.

$E \in \mathcal{A}$: σ-semistable

$\Longrightarrow \phi(E[p]) := \phi(E) + p$.

Prop

① (Macri)

$Q = K_l$: l-Kronecker quiver
$$\begin{tikzcd} 1 \arrow[r, bend left, "a_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "a_l"'] & 2 \end{tikzcd}$$

$^{\forall}\sigma \in \mathrm{Stab}(D^b(K_\ell))$, $^{\exists}$ full σ-exc. coll.

② (Dimitrov – Katzarkov).

$Q = A_{1.2}^{(1)}$: affine $A_{1.2}$ quiver

$$\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}$$

$^{\forall}\sigma \in \mathrm{Stab}(D^b(A_{1.2}^{(1)}))$, $^{\exists}$ full σ-exc. coll.

<u>Conj</u> (Dimitrov – Katzarkov).

$Q = \vec{\Delta}$: Dynkin quiver.

$^{\forall}\sigma \in \mathrm{Stab}(D^b(\vec{\Delta}))$, $^{\exists}$ full σ-exc. coll.

<u>Thm</u> (O)

Conj by Dimitrov – Katzarkov is true!

In particular, $^{\forall}\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(D^b(\vec{\Delta}))$,

$^{\exists}$ full σ-exc. coll. $\mathcal{E}$

s.t. $\quad \mathcal{A} \simeq \langle \mathcal{E} \rangle_{ex}$.

(Idea of proof).

① (King - Qiu).

Any heart in $D^b(\vec{\Delta})$ can be obtained from the std heart $\mathrm{mod}(\mathbb{C}\vec{\Delta})$ by iterated simple tilts.

② Simple objects in a given heart form a full Ext-exc. coll.

§3. Space of stability conditions

$\mathcal{D} = D^b(\vec{\Delta})$, $\vec{\Delta}$: Dynkin quiver

$\chi : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \longrightarrow \mathbb{Z}$: Euler form.

$$\chi(E, F) := \sum_{p \in \mathbb{Z}} (-1)^p \dim_{\mathbb{C}} \mathrm{Hom}(E, F[p]).$$

$I := \chi + \chi^T$: Cartan form.

$$\Delta_{re} := \{ [E] \in K_0(\mathcal{D}) \mid E \text{ is indecomposable} \}$$

$$\underset{\uparrow}{=} \{ [E] \in K_0(\mathcal{D}) \mid E \text{ is exc.} \}.$$

only Dynkin case.

$c := -[S] \in \mathrm{Aut}(K_0(\mathcal{D}), I)$: Coxeter transformation

Prop

$(K_0(D^b(\vec{\Delta})), I, \Delta_{re})$ is a root system of type corresponding to $\vec{\Delta}$.

Moreover, $(K_0(D^b(\vec{\Delta})), I, \Delta_{re}, c)$ is a generalized root system in the sense of K. Saito

$\mathfrak{h} := \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$: Cartan sub algebra

W : Weyl gp assoc. to. $(K_0(\mathcal{D}), I, \Delta_{re})$.

$W \curvearrowright K_0(\mathcal{D}) \rightsquigarrow W \curvearrowright \mathfrak{h}$.

By Chevalley's Thm,

$$\mathfrak{h}/W \cong \mathbb{C}^{\mu} \text{ as cpx mfds.}$$

where $\mu = \#\text{Vertex of } \vec{\Delta} = \mathrm{rk}_{\mathbb{Z}} K_0(D^b(\vec{\Delta}))$.

Conj (Takahashi).

$$\boxed{\mathfrak{h}/W \cong \mathrm{Stab}(D^b(\vec{\Delta}))} \quad \text{as cpx mfds.}$$

Conj is confirmed when

$\vec{\Delta} = A_2$: Bridgeland – Qiu – Sutherland,

$\vec{\Delta} = A_\mu$: Haiden – Katzarkov – Kontsevich

$Q = A_{p,q}^{(1)}$: "

($Q = K_\ell$: Dimitrov – Katzarkov,
Ikeda – O – Shiraishi – Takahashi)

Moreover, it is expected that

$$\begin{tikzcd} \mathfrak{h}/W \arrow[rr, "\sim"] \arrow[dr, "\Pi_\zeta"'] & & \mathrm{Stab}(D^b(\vec{\Delta})) \arrow[dl] \\ & \mathrm{Hom}(K_0(D^b(\vec{\Delta})), \mathbb{C}) \ni Z & \end{tikzcd}$$

$$\mathrm{Stab}(D^b(\vec{\Delta})) \ni (Z, \mathcal{A}) \longmapsto Z$$

Here, $\Pi_\zeta : \mathfrak{h}/W \longrightarrow \mathrm{Hom}(K_0(D^b(\vec{\Delta})), \mathbb{C})$ is the exponential period mapping associated to a primitive form ζ.

Remark

Π_ζ depends on $(K_0(D^b(\vec{\Delta})), I, \Delta^{re}, \underset{\sim}{c})$.

It is know that $\exists\, \sigma_0 \in \mathrm{Stab}(D^b(\vec{\Delta}))$ s.t. σ_0 corresponds to $0 \in \mathbb{C}^\mu \cong \mathfrak{h}/W$.

Prop (Kajiura – Saito – Takahashi, O – Takahashi).

$\exists\, \sigma_0 = (Z_0, \mathcal{A}_0) \in \mathrm{Stab}(D^b(\vec{\Delta}))$

s.t. Z_0 = exp. period map assoc. to ζ.

$\mathcal{A}_0 \cong \mathrm{mod}(\mathbb{C}\underset{\sim}{\vec{\Delta}_P})$.

Dynkin quiver w/ principal orientation

$\exists\, \mathcal{E}_0 := (S_1, \cdots, S_\mu)$: full Ext-exc. coll.

s.t. $\langle \mathcal{E}_0 \rangle_{ex} \cong \mathrm{mod}\, \mathbb{C}\vec{\Delta}_P \cong \mathcal{A}_0$.

Prop

$Br_\mu :=$ μ-stands Artin's braid gp.

$FEC(D^b(\vec{\Delta})) :=$ the set of isom. classes of f.e.c. in $D^b(\vec{\Delta})$.

$Br_\mu \ltimes \mathbb{Z}^\mu \curvearrowright FEC(D^b(\vec{\Delta}))$ as mutations and shifts.

This action is transitive.

Conj

$\forall s \in \mathbb{C}^\mu \cong \mathfrak{h}/W$, $\exists g_s \in Br_\mu \ltimes \mathbb{Z}^\mu$

s.t. ① $\mathcal{E}_s := g_s \cdot \mathcal{E}_0$ is a full Ext-exc. coll.

② $Z_s : \langle \mathcal{E}_s \rangle_{ex} \longrightarrow \mathbb{C}$ defined by the exp. period map assoc. to ζ is a stability function.

In particular, $(Z_s, \mathcal{A}_s)$ is a stability condition.